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Large Deflection of Sandwich Plates with Orthotropic Cores

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Nomenclature

x, y, z	=	rectangular coordinates
a, b	=	length and width of sandwich plate
h	=	thickness of core
t	=	thickness of face layers
E, ν	=	Young's modulus of elasticity and Poisson's ratio
G_{xz}, G_{yz}	=	shear moduli of core
τ_{xz}, τ_{yz}	=	stresses in core
q	=	load per unit area
M_x, M_y, M_{xy}	=	bending and twisting moments per unit length
N_x, N_y, N_{xy}	=	stress resultants in middle plane of face layers per unit length
Q_x, Q_y	=	shear forces per unit length
u, v, w	=	displacements in x, y, and z directions
$\beta, \beta', \gamma, \gamma'$	=	generalized boundary displacements
$\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5$	=	Lagrangian multipliers

Introduction

THE problem of large deflection of sandwich plates has been investigated by several authors. Reissner[3] presented an exact analysis of finite deflections of sandwich plates, Wang[4] gave a general theory of large deflection of homogeneous and sandwich plates and shells, Hoff[5] and Eringen[6] each developed a theory of bending and buckling of sandwich plates. In all of the foregoing investigations the core and facings of the sandwich plates were assumed to be isotropic. In the present analysis the core is taken as an orthotropic honeycomb-type structure. It is felt that this type of core corresponds more exactly to the behavior of actual sandwich construction used in industry.

The sandwich plate shown in Fig. 1 is assumed to consist of two thin isotropic face layers, each, of thickness t separated by and bonded to an orthotropic core of thickness h. The usual assumptions for sandwich plates are adopted here, as in Ref. 3. In addition, the effect of the transverse normal stresses in the core is considered negligibly small compared with the effect of the transverse shear stresses on the over-all behavior of the plate.

Analysis

Since all the face-parallel core stresses are neglected, the face-parallel stress resultants of the composite plate are due to the stresses in the face layers only as shown in Fig. 2 and may be obtained as follows:

$$N_x = N_{xu} + N_{xl} \qquad N_y = N_{yu} + N_{yl}$$
$$N_{xy} = N_{xyu} + N_{xyl}$$

where the subscripts u and l refer to the upper and lower face layers, respectively.

The differential equations of equilibrium are[1]

$$\left.\begin{aligned} &\frac{\partial N_x}{\partial x} + \frac{\partial N_{xy}}{\partial y} = 0 \qquad \frac{\partial N_y}{\partial y} + \frac{\partial N_{xy}}{\partial x} = 0 \\ &\frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} + q + N_x\frac{\partial^2 w}{\partial x^2} + N_y\frac{\partial^2 w}{\partial y^2} + 2N_{xy}\frac{\partial^2 w}{\partial x \partial y} = 0 \\ &\frac{\partial M_x}{\partial x} - \frac{\partial M_{xy}}{\partial y} - Q_x = 0 \\ &\frac{\partial M_y}{\partial y} - \frac{\partial M_{xy}}{\partial x} - Q_y = 0 \end{aligned}\right\} \quad (1)$$

The five foregoing equations contain eight unknowns, N_x, N_y, N_{xy}, M_x, M_y, M_{xy}, Q_x, and Q_y. More equations will be obtained through the use of the variational theorem of complementary energy in conjunction with Lagrangian multipliers.

The strain energy of the two face layers is given by the following expression:

$$V_f = \frac{1}{2}\iint \frac{1}{2Et}\left[N_x^2 + N_y^2 - 2\nu N_x N_y + 2(1+\nu)N_{xy}^2\right] + \frac{2}{Et(h+t)^2}\left[M_x^2 + M_y^2 - 2\nu M_x M_y + 2(1+\nu)M_{xy}^2\right] + \left[N_x\left(\frac{\partial w}{\partial x}\right)^2 + N_y\left(\frac{\partial w}{\partial y}\right)^2 + 2N_{xy}\frac{\partial w}{\partial x}\frac{\partial w}{\partial y}\right]\Bigg\}\, dx\, dy$$

The strain energy stored in the core is given by

$$V_c = \frac{1}{2h}\iint\left[\frac{Qx^2}{G_{xz}} + \frac{Qy^2}{G_{yz}}\right] dx\, dy$$

The work done by the surface forces over that portion of the surface where the displacements are prescribed is given by

$$W = \int_{-b/2}^{b/2}\left[N_x u + N_{xy} v + \left(N_x\frac{\partial w}{\partial x} + N_{xy}\frac{\partial w}{\partial y} + Q_x\right)\times w + M_x\beta + M_{xy}\gamma'\right]_{x=a/2,\ x=-a/2} dy + \int_{-a/2}^{a/2}\left[N_y v + N_{xy} u + \left(N_y\frac{\partial w}{\partial y} + N_{xy}\frac{\partial w}{\partial x} + Q_y\right)w + M_y\gamma + M_{xy}\beta'\right]_{y=b/2,\ y=-b/2} dx$$

In order to render the complementary energy $(V_f + Vc - W)$ a minimum subject to the equations of equilibrium (1), these equations are multiplied by Lagrangian multipliers λ_1, λ_2, λ_3, λ_4, and λ_5, respectively, and then integrated over the area of the plate. This result is added to the complementary energy, the first variation of the resulting expression is carried out with respect to the unknown functions N_x, N_y, N_{xy}, M_x, M_y, M_{xy}, Q_x, and Q_y, and the result is set equal to zero:

$$\delta V_f + \delta V_c - \delta W + \delta\iint\left[\lambda_1\left(\frac{\partial N_x}{\partial x} + \frac{\partial N_{xy}}{\partial y}\right) + \lambda_2\left(\frac{\partial N_y}{\partial y} + \frac{\partial N_{xy}}{\partial x}\right) + \lambda_3\left(\frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} + q + N_x\frac{\partial^2 w}{\partial x^2} + N_y\frac{\partial^2 w}{\partial y^2} + 2N_{xy}\frac{\partial^2 w}{\partial x\partial y}\right) + \lambda_4\left(\frac{\partial M_x}{\partial x} - \frac{\partial M_{xy}}{\partial y} - Q_x\right) + \lambda_5\left(\frac{\partial M_y}{\partial y} - \frac{\partial M_{xy}}{\partial x} - Q_y\right)\right] dxdy = 0$$

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Substituting for V_f, V_c, and W the forementioned expressions and using integration by parts to eliminate the derivatives of the variations of the unknown functions, we obtain the variational equation that gives the relations between the Lagrangian multiplier and the generalized displacements as well as the so-called Euler equations, which are equivalent to the required stress-strain relationships:

$$\lambda_1 = u \qquad \lambda_2 = v \qquad \lambda_3 = w$$
$$\lambda_4 = \beta = -\beta' \qquad \varphi_5 = \gamma = -\gamma'$$

$$\frac{N_x - \nu N_y}{2Et} = \frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 \tag{2}$$

$$\frac{N_y - \nu N_x}{2Et} = \frac{\partial v}{\partial y} + \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2 \tag{3}$$

$$\frac{(1+\nu)N_{xy}}{Et} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial w}{\partial x}\frac{\partial w}{\partial y} \tag{4}$$

$$M_x - \nu M_y = D(1-\nu^2)\frac{\partial \beta}{\partial x} \tag{5}$$

$$M_y - \nu M_x = D(1-\nu^2)\frac{\partial \gamma}{\partial y} \tag{6}$$

$$M_{xy} = -D\left(\frac{1-\nu}{2}\right)\left(\frac{\partial \beta}{\partial y} + \frac{\partial \gamma}{\partial x}\right) \tag{7}$$

$$\beta = -\frac{\partial w}{\partial x} + \frac{Q_x}{hG_{xz}} \tag{8}$$

$$\gamma = -\frac{\partial w}{\partial y} + \frac{Q_y}{hG_{yz}} \tag{9}$$

where

$$D = \frac{Et(h+t)^2}{2(1-\nu^2)}$$

Equations (2–9), together with the equilibrium equations (1), constitute the 13 basic differential equations for the finite deflection of sandwich plates with orthotropic cores. It is possible to reduce these equations to two nonlinear partial differential equations in terms of the transverse deflection w and an Airy stress function F. This result would represent a generalization of the fundamental equations governing the finite of deflection of thin homogeneous plates developed by von Kármán and given by Timoshenko (Ref. 1, p. 417).

Derivation of the Two Simultaneous Differential Equations

The first equation is obtained by differentiating Eq. (2) once with respect to y and Eq. (4) once with respect to x and

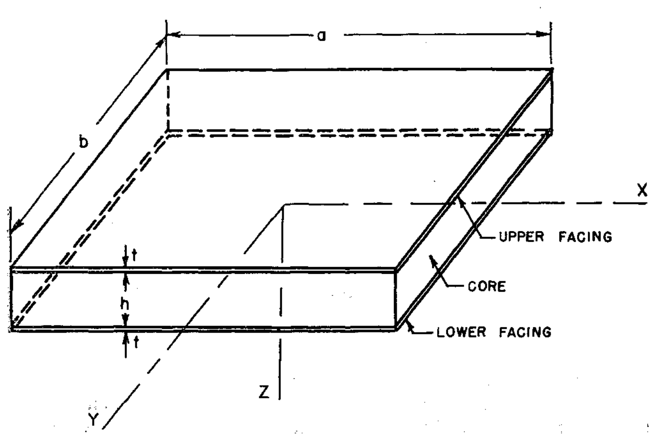

Fig. 1 Rectangular sandwich plate.

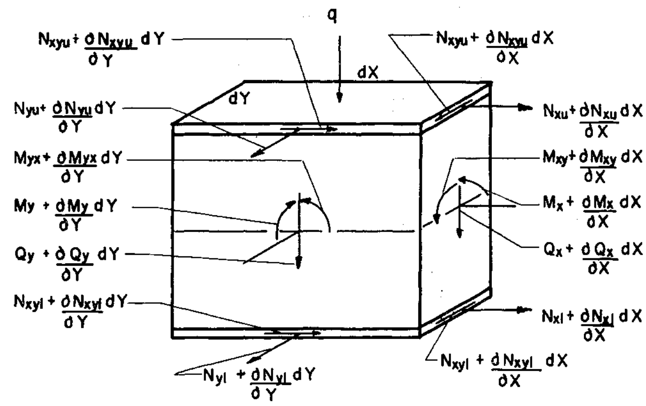

Fig. 2 Element of composite plate showing resultant forces and moments.

subtracting one from the other. Then differentiate the resulting equation once with respect to y and Eq. (3) twice with respect to x and add. Introduction of Airy's stress function F, defined as

$$N_x - \frac{\partial^2 F}{\partial y^2} \qquad N_y = \frac{\partial^2 F}{\partial x^2} \qquad N_{xy} = -\frac{\partial^2 F}{\partial x \partial y}$$

gives the first equation in its final form:

$$\Delta\Delta F = 2Et\left[\left(\frac{\partial^2 w}{\partial x \partial y}\right)^2 - \frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2}\right] \tag{10}$$

Equation (10) is exactly the same as for thin homogeneous plates.[1]

The second equation is derived by complicated and rather lengthy processes of substitution and differentiation.[7, 2] However, the second equation in its final form is as follows:

$$\left(1 - D_y\frac{\partial^2}{\partial x^2} - D_x\frac{\partial^2}{\partial y^2}\right)\Delta\Delta w =$$
$$\frac{1}{D}\left[1 - \left(D_y + \frac{2D_x}{1-\nu}\right)\frac{\partial^2}{\partial x^2} - \left(D_x + \frac{2D_y}{1-\nu}\right)\frac{\partial^2}{\partial y^2} + \right.$$
$$\left.\frac{2D_xD_y}{1-\nu}\Delta\Delta\right]\left(q + \frac{\partial^2 F}{\partial y^2}\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 F}{\partial x^2}\frac{\partial^2 w}{\partial y^2} - 2\frac{\partial^2 F}{\partial x \partial y}\frac{\partial^2 w}{\partial x \partial y}\right) \tag{11}$$

where

$$D_x = \frac{(1-\nu)D}{2hG_{xz}} \qquad D_y = \frac{(1-\nu)D}{2hG_{yz}}$$

In the case of homogeneous plates ($G_{xz} = G_{yz} = \infty$ or $D_x = D_y = 0$), Eqs. (10) and (11) reduce to the two well-known equations [(245) and (246) of Ref. 1, p. 417] for large deflections of thin homogeneous plates.

Equations (10) and (11) represent the governing differential equation for finite deflection of sandwich plates with orthotropic cores. They can be solved simultaneously for any rectangular plate with specific boundary conditions, using the same techniques as in the case of finite deflections of thin homogeneous plates.[1]

For detailed numerical solution for large deflection of simply-supported rectangular sandwich plates with isotropic cores, see Ref. 7.

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Stress Functions for the Axisymmetric, Orthotropic, Elasticity Equations

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I. Introduction

SOUTHWELL expressed the axisymmetric elastic field equations for an isotropic body in terms of two stress functions.[1] The Southwell stress functions have found considerable application in the structural analyses of solid rocket motors.[2–4] These analyses, however, are only applicable to motors constructed of isotropic propellants, and, thus, some recent proposed motor designs that make use of orthotropic propellants give rise to new unsolved problems. So that similar analysis techniques may be applied to orthotropic grains as were applied to isotropic grains, it is desirable to express the problem in terms of stress functions of a nature similar to the Southwell functions. In particular, such functions should reduce to those of Southwell's for the particular case of isotropy.

II. Theory

The linear elastic equations expressed in cylindrical coordinates for a cylindrical orthotropic axisymmetric body subjected to axisymmetric loads are

$$\frac{\partial \tau_{rr}}{\partial r} + \frac{\partial \tau_{rz}}{\partial z} + \frac{\tau_{rr} - \tau_{\theta\theta}}{r} + F_r = 0 \tag{1}$$

$$\frac{\partial \tau_{rz}}{\partial r} + \frac{\partial \tau_{zz}}{\partial z} + \frac{\tau_{rz}}{r} + F_z = 0 \tag{2}$$

$$\epsilon_{rr} = \frac{\partial u_r}{\partial r} = S_{11}\tau_{rr} + S_{12}\tau_{\theta\theta} + S_{13}\tau_{zz} + e_1 \tag{3}$$

$$\epsilon_{\theta\theta} = u_r/r = S_{12}\tau_{rr} + S_{22}\tau_{\theta\theta} + S_{23}\tau_{zz} + e_2 \tag{4}$$

$$\epsilon_{zz} = \partial u_z/\partial z = S_{13}\tau_{rr} + S_{23}\tau_{\theta\theta} + S_{33}\tau_{zz} + e_3 \tag{5}$$

$$\epsilon_{rz} = \frac{1}{2}\left(\frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r}\right) = \frac{S_{44}}{2}\tau_{rz} \tag{6}$$

Equations (1) and (2) are the equilibrium equations, and Eqs. (3–6) are the stress (τ_{ij}) strain (ϵ_{ij}) law for a cylindrical orthotropic material[5] (displacement components are denoted by u_i and the orthotropic elastic constants by S_{ij}). The free expansion in the i direction caused by a temperature change $\Delta T = T - T_0$ is given by e_i, i.e.,

$$e_i = \int_{T_0}^{T} \alpha_i dT'$$

The instantaneous coefficient of linear expansion is denoted by α_i, i.e., $\alpha_i = (1/L_i)(dL_i/dT)$.

A stress function $\psi(r,z)$ is defined by the following equation:

$$\tau_{zz} = -(1/r)(\partial\psi/\partial r) \tag{7}$$

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A function $f_z(r,z)$ related to the axial body force $F_z(r,z)$ is defined by the following equation:

$$F_z = (1/r)(\partial f_z/\partial r) \tag{8}$$

Introducing Eqs. (7) and (8) into Eq. (2) gives

$$\frac{\partial}{\partial r}(r\tau_{rz}) - \frac{\partial}{\partial r}\left(\frac{\partial\psi}{\partial z}\right) + \frac{\partial f_z}{\partial r} = 0 \tag{9}$$

Integration of Eq. (9) yields (it may be simply shown that the function of integration may be set equal to zero without loss of generality)

$$\tau_{rz} = \frac{1}{r}\frac{\partial\psi}{\partial z} - \frac{1}{r}f_z \tag{10}$$

Introducing Eq. (7) into Eqs. (3) and (4) and solving for τ_{rr} and $\tau_{\theta\theta}$, the following expressions are found:

$$\tau_{\theta\theta} = \frac{S_{12}\dfrac{\partial u_r}{\partial r} - S_{11}\dfrac{u_r}{r} + (S_{12}S_{13} - S_{11}S_{23})\dfrac{1}{r}\dfrac{\partial\psi}{\partial r} - S_{12}e_1 + S_{11}e_2}{(S_{12})^2 - S_{11}S_{22}} \tag{11}$$

$$\tau_{rr} = \frac{S_{12}\dfrac{u_r}{r} - S_{22}\dfrac{\partial u_r}{\partial r} + (S_{23}S_{12} - S_{22}S_{13})\dfrac{1}{r}\dfrac{\partial\psi}{\partial r} + S_{22}e_1 - S_{12}e_2}{(S_{12})^2 - S_{11}S_{22}} \tag{12}$$

The problem could now be expressed in terms of the stress function ψ and the displacement component u_r; however, in order to obtain a formulation that reduces to the Southwell formulation (for an isotropic material), it is necessary to express u_r in terms of ψ and a new function φ†:

$$u_r = \frac{S_{11}S_{22} - (S_{12})^2}{S_{22}}\frac{\varphi}{r} + \frac{S_{22}(S_{44} + S_{13} - S_{11}) + S_{12}(S_{12} - S_{23})}{S_{22}}\frac{\psi}{r} \tag{13}$$

Introducing Eq. (13) into Eqs. (11) and (12) and making the definitions

$$d_1 = (S_{12})^2 - S_{11}S_{22} \tag{14}$$

$$d_2 = -\left[1 + \frac{1}{d_1}(S_{22}S_{44} + S_{22}S_{13} - S_{12}S_{23})\right] \tag{15}$$

$$d_3 = -\left[1 + \frac{1}{d_1}(S_{22}S_{44} + 2S_{22}S_{13} - 2S_{12}S_{23})\right] \tag{16}$$

$$d_4 = -\left[\frac{S_{12}}{S_{22}}\left(1 - \frac{S_{12}S_{23}}{d_1}\right) + \frac{S_{12}(S_{44} + 2S_{13}) - S_{11}S_{23}}{d_1}\right] \tag{17}$$

the expressions for τ_{rr} and $\tau_{\theta\theta}$ are found to be

$$\tau_{rr} = \frac{1}{r}\left[\frac{\partial\varphi}{\partial r} + d_3\frac{\partial\psi}{\partial r}\right] - \left(1 + \frac{S_{12}}{S_{22}}\right)\frac{1}{r^2}[d_2\psi + \varphi] + \frac{S_{22}e_1 - S_{12}e_2}{d_1} \tag{18}$$

$$\tau_{\theta\theta} = -\frac{1}{r}\left[\frac{S_{12}}{S_{22}}\frac{\partial\varphi}{\partial r} + d_4\frac{\partial\psi}{\partial r}\right] + \frac{(S_{12} + S_{11})}{S_{22}}\frac{1}{r^2}[d_2\psi + \varphi] + \frac{S_{11}e_2 - S_{12}e_1}{d_1} \tag{19}$$

† There are many different relationships between u_r, φ, and ψ which would lead to the desired results. The particular one used herein was selected in an attempt to simplify the resulting equations as much as possible.